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Nonlinear (4, 8, 4) Multiplet of $\mathcal{N}{=}8, d{=}1$ Supersymmetry

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Abstract

We construct a nonlinear version of the $d{=}1$ off-shell $\mathcal{N}{=}8$ multiplet $(\mathbf{4},\mathbf{8},\mathbf{4})$, proceeding from a nonlinear realization of the superconformal group $OSp(4^\star|4)$ in the $\mathcal{N}{=}8\,, d{=}1$ analytic bi-harmonic superspace. The new multiplet is described by a double-charged analytic superfield $q^{1,1}$ subjected to some nonlinear harmonic constraints which are covariant under the $OSp(4^\star|4)$ transformations. Together with the analytic superspace coordinates, $q^{1,1}$ parametrizes an analytic coset manifold of $OSp(4^\star|4)$ and so is a Goldstone superfield. In any $q^{1,1}$ action the superconformal symmetry is broken, while $\mathcal{N}{=}8\,, d{=}1$ Poincaré supersymmetry can still be preserved. We construct the most general class of such supersymmetric actions and find the general expression for the bosonic target metric in terms of the original analytic Lagrangian superfield density which is thus the target geometry prepotential. It also completely specifies the scalar potential. The metric is conformally flat and, in the $SO(4)$ invariant case, is a deformation of the metric of a four-sphere S^4.

1 Introduction

Extended d=1 supersymmetry reveals a number of specific features which are not shared by its higher-dimensional counterparts. One of such peculiarities is the existence of nonlinear cousins of the standard linear d=1 supermultiplets. They possess the same off-shell component contents, but are described by d=1 superfields subjected to some nonlinear constraints. As a result, the relevant d=1 supersymmetries are realized on these multiplets by intrinsically nonlinear transformations. While the target geometries of supersymmetric d=1 sigma models associated with linear multiplets were studied in detail [1], no such an exhaustive study was undertaken so far for the nonlinear ones. Also, it is an open question which realistic physical systems of supersymmetric quantum mechanics (SQM) can be described by d=1 models based on nonlinear multiplets and whether the latter can be recovered within some modified dimensional reduction procedure. To answer these and related questions, it is important to have more examples of the nonlinear multiplets and supersymmetric models constructed on their basis.

In the $\mathcal{N}$=4, d=1 case nonlinear versions are known for the off-shell multiplets $(\mathbf{3},\mathbf{4},\mathbf{1})$ [2, 3], $(\mathbf{2},\mathbf{4},\mathbf{2})$ [3] and $(\mathbf{4},\mathbf{4},\mathbf{0})$ [4, 5, 6, 7]. The detailed structure of the corresponding invariant actions and interrelations between various multiplets were studied in refs. [8]-[10]. In the case of $\mathcal{N}$=8, d=1 supersymmetry the only nonlinear multiplet known so far is the appropriate counterpart of the chiral $\mathcal{N}$=8 multiplet $(\mathbf{2},\mathbf{8},\mathbf{6})$ [11].

In this paper we describe one more nonlinear $\mathcal{N}$=8 multiplet, a counterpart of the linear multiplet $(\mathbf{4},\mathbf{8},\mathbf{4})$ studied in [12, 13]. We make use of the manifestly $\mathcal{N}$=8 supersymmetric language of the bi-harmonic $\mathcal{N}$=8, d=1 superspace [13]. The $\mathcal{N}$=4 nonlinear multiplets $(\mathbf{3},\mathbf{4},\mathbf{1})$ and $(\mathbf{2},\mathbf{4},\mathbf{2})$ were deduced in [3] as the Goldstone multiplets associated with nonlinear realizations of the most general $\mathcal{N}$=4, d=1 superconformal group $D(2,1;\alpha)$. The nonlinear $(\mathbf{4},\mathbf{8},\mathbf{4})$ multiplet is also Goldstone multiplet, this time associated with a coset of one of the d=1 superconformal supergroups [14], the supergroup $OSp(4^\star|4)$. The power of the $\mathcal{N}$=8 bi-harmonic superspace manifests itself in the fact that there is no need to use the standard routine of nonlinear realizations, i.e. to start from the explicit structure relations of $OSp(4^\star|4)$, to construct the relevant Cartan forms, etc. The $\mathcal{N}$=8 multiplet in question is described by the analytic bi-harmonic superfield of the same type as in the case of linear $(\mathbf{4},\mathbf{8},\mathbf{4})$ multiplet [13], but this superfield is now subjected to some nonlinear harmonic constraints. The precise form of these constraints is almost uniquely determined from the requirement of covariance under nonlinear $SO(5)/SO(4)$ transformations (belonging to $OSp(4^\star|4)$) and nonlinear version of $\mathcal{N}$=8, d=1 Poincaré supersymmetry. The most characteristic novel feature of these and other transformations of the supergroup $OSp(4^\star|4)$ on the analytic superspace coordinates is that they non-trivially mix these coordinates with the Goldstone analytic superfield.

In Sect. 2 we recollect the pivotal features of the $\mathcal{N}$=8, d=1 bi-harmonic superspace and the superfield description of the linear $(\mathbf{4},\mathbf{8},\mathbf{4})$ multiplet in it. In Sect. 3 we define the nonlinear $(\mathbf{4},\mathbf{8},\mathbf{4})$ multiplet within the same off-shell superfield framework. We explicitly give the analytic superspace form of the basic $OSp(4^\star|4)$ transformations containing in their closure all other transformations. The simplest invariant action quadratic in the $(\mathbf{4},\mathbf{8},\mathbf{4})$ superfield is constructed in Sect. 4. It already yields a non-trivial nonlinear sigma model in components. It is a deformation of the $SO(5)/SO(4)$ nonlinear sigma model. The action respects the invariance only under some subgroup of $OSp(4^\star|4)$ involving $\mathcal{N}$=8, d=1 Poincaré symmetry. It is impossible to set up the $OSp(4^\star|4)$ invariant action in this case at all, because of lacking of dilaton among the components of the nonlinear $(\mathbf{4},\mathbf{8},\mathbf{4})$ multiplet. In Sect. 4 we also construct the most general action of this multiplet invariant under $\mathcal{N}$=8, d=1 Poincaré supersymmetry and study its bosonic sector. We derive the nonlinear differential equation satisfied by the target bosonic metric and show that our bi-harmonic formalism supplies a natural general solution for this metric in terms of the analytic superfield Lagrangian density which thus plays the role of the prepotential of the relevant target geometry. It also completely determines the form of the admissible scalar potential.

2 Bi-harmonic superspace and linear $\mathcal{N}$=8, d=1 multiplet (4,8,4)

2.1 Standard and bi-harmonic $\mathcal{N}$=8 superspaces

The standard real $\mathcal{N}$=8 , d=1 superspace is defined as the following set of coordinates:

$$\mathbf{R}^{(1|8)} = (\,Z\,) = (\,t\,, \theta^{i\underline{k}}\,, \theta^{a\underline{b}}\,)\,.$$

Here the indices i , $\underline{k}$ and a , $\underline{b}$ are doublet indices of four commuting $SU(2)$ groups forming the subgroup $SO(4)\times SO(4)$ of the full automorphism group $SO(8)$ of $\mathcal{N}$=8, d=1 Poincaré superalgebra. The covariant spinor derivatives are defined as

$$\begin{aligned}&D_{i\underline{k}} = \frac{\partial}{\partial\theta^{i\underline{k}}} + i\,\theta_{i\underline{k}}\,\partial_t\,, \quad D_{a\underline{b}} = \frac{\partial}{\partial\theta^{a\underline{b}}} + i\,\theta_{a\underline{b}}\,\partial_t\,,\\ &(D_{i\underline{k}})^{\dagger} = -\varepsilon^{i\,l}\,\varepsilon^{\underline{k}\,\underline{n}}\,D_{l\,\underline{n}}\,, \quad (D_{a\underline{b}})^{\dagger} = -\varepsilon^{a\,c}\,\varepsilon^{\underline{b}\,\underline{d}}\,D_{c\,\underline{d}}\,,\end{aligned} \tag{2.1}$$

and obey the following algebra:

$$\{\,D_{i\underline{k}}\,, D_{j\,\underline{l}}\,\} = 2i\,\varepsilon_{i\,j}\,\varepsilon_{\underline{k}\,\underline{l}}\,\partial_t\,, \quad \{\,D_{a\underline{b}}\,, D_{c\underline{d}}\,\} = 2i\,\varepsilon_{a\,c}\,\varepsilon_{\underline{b}\,\underline{d}}\,\partial_t\,. \tag{2.2}$$

For the one–dimensional $\mathcal{N}$=8 supersymmetric theory we can introduce $SU(2)\times SU(2)$ bi-harmonic superspace (HSS) with two independent sets of harmonic variables $u^{\pm1}_i$ and $v^{\pm1}_a$ associated with two different $SU(2)$ groups of the above $SO(4)\times SO(4)$ automorphism group. As shown in [13], it provides the appropriate framework for the $\mathcal{N}$=8 supersymmetric quantum mechanics associated with the d=1 off-shell supermultiplet $(\mathbf{4},\mathbf{8},\mathbf{4})$ [17].

We define the central basis of this HSS as

$$\mathbf{HR}^{(1+2+2|8)} = (\,Z\,, u\,, v\,) = \mathbf{R}^{(1|8)} \otimes (\,u^{\pm1}_i, v^{\pm1}_a\,)\,, \quad u^{1i}u^{-1}_i = 1\,, \quad v^{1a}v^{-1}_a = 1\,. \tag{2.3}$$

The analytic basis in the same bi-harmonic SS amounts to the following choice of coordinates:

$$\mathbf{HR}^{(1+2+2|8)} = (\,X\,, u\,, v\,) = (\,t_A\,, \theta^{\pm1,0\,\underline{i}}\,, \theta^{0,\pm1\,\underline{a}}\,, u^{\pm1}_i\,, v^{\pm1}_a\,) \tag{2.4}$$

where

$$t_A = t + i(\theta^{1,0\,\underline{i}}\,\theta^{-1,0}_{\underline{i}} + \theta^{0,1\,\underline{a}}\,\theta^{0,-1}_{\underline{a}})\,, \quad \theta^{\pm1,0\,\underline{i}} = \theta^{k\,\underline{i}}\,u^{\pm1}_k\,, \quad \theta^{0,\pm1\,\underline{a}} = \theta^{b\,\underline{a}}\,v^{\pm1}_b\,.$$

The analytic basis makes manifest the existence of the *analytic subspace* in the bi-harmonic SS

$$\mathbf{AR}^{(1+2+2|4)} = (\,\zeta\,, u\,, v\,) = (\,t_A\,, \theta^{1,0\,\underline{i}}\,, \theta^{0,1\,\underline{a}}\,, u^{\pm1}_i\,, v^{\pm1}_a\,)\,, \tag{2.5}$$

which has twice as less odd coordinates as compared to the standard $\mathcal{N}$=8, d=1 superspace and is closed under $\mathcal{N}$=8 supersymmetry transformations

$$\delta t_A = 2i\left(\epsilon^{-1,0\,\underline{i}}\theta^{1,0}_{\underline{i}} + \epsilon^{0,-1\,\underline{a}}\theta^{0,1}_{\underline{a}}\right), \quad \delta\theta^{1,0\,\underline{i}} = \epsilon^{1,0\,\underline{i}}, \ \delta\theta^{0,1\,\underline{a}} = \epsilon^{0,1\,\underline{a}}, \tag{2.6}$$

where $\epsilon^{\pm1,0\,\underline{i}} = \epsilon^{i\underline{i}}u^{\pm1}_i$, etc. The existence of the analytic subspace matches the form of covariant spinor derivatives in the analytic basis

$$D^{1,0\,\underline{i}} = \frac{\partial}{\partial\theta^{-1,0}_{\underline{i}}}, \quad D^{0,1\,\underline{a}} = \frac{\partial}{\partial\theta^{0,-1}_{\underline{a}}} \tag{2.7}$$

where

$$D^{\pm1,0\,\underline{i}} \equiv D^{k\,\underline{i}}\,u^{\pm1}_k, \quad D^{0,\pm1\,\underline{a}} \equiv D^{b\,\underline{a}}\,v^{\pm1}_b\,. \tag{2.8}$$

The "shortness" of $D^{1,0\,\underline{i}}, D^{0,1\,\underline{a}}$ means that the analytic bi-harmonic superfields $\Phi^{\,q,p}$,

$$D^{1,0\,\underline{i}}\,\Phi^{\,q,p} = D^{0,1\,\underline{a}}\,\Phi^{\,q,p} = 0\,, \tag{2.9}$$

do not depend on $\theta^{-1,0\,\underline{i}}, \theta^{0,-1\,\underline{a}}$ in the analytic basis, i.e. "live" on the analytic subspace (2.5):

$$\Phi^{\,q,p} = \Phi^{\,q,p}(\zeta, u, v)\,. \tag{2.10}$$

In the bi-harmonic superspace one can define two sets of mutually commuting harmonic derivatives, each forming an $SU(2)$ algebra [15, 13]. In the analytic basis and being applied to the analytic superfields, the derivatives with positive $U(1)$ charges, as well as the derivatives counting the harmonic $U(1)$ charges p, q, read

$$\begin{aligned}
D^{2,0} &= \partial^{2,0} + i\,\theta^{1,0\,\underline{i}}\theta^{1,0}_{\underline{i}}\partial_t\,, \quad D^{0,0}_u = \partial^{0,0}_u + \theta^{1,0\,\underline{i}}\frac{\partial}{\partial\theta^{1,0\,\underline{i}}}\,,\\
D^{0,2} &= \partial^{0,2} + i\,\theta^{0,1\,\underline{a}}\theta^{0,1}_{\underline{a}}\partial_t\,, \quad D^{0,0}_v = \partial^{0,0}_v + \theta^{0,1\,\underline{a}}\frac{\partial}{\partial\theta^{0,1\,\underline{a}}}\,,
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
\partial^{2,0} &= u^{1i}\frac{\partial}{\partial u^{-1i}}\,, \quad \partial^{0,0}_u = u^{1i}\frac{\partial}{\partial u^{1i}} - u^{-1i}\frac{\partial}{\partial u^{-1i}}\,,\\
\partial^{0,2} &= v^{1a}\frac{\partial}{\partial v^{-1a}}\,, \quad \partial^{0,0}_v = v^{1a}\frac{\partial}{\partial v^{1a}} - v^{-1a}\frac{\partial}{\partial v^{-1a}}\,.
\end{aligned} \tag{2.12}$$

2.2 The linear multiplet (4,8,4)

In the standard $\mathcal{N}$=8, d=1 superspace $\mathbf{R}^{(1|8)}$ the multiplet with the off-shell field content $(\mathbf{4}, \mathbf{8}, \mathbf{4})$ is described by a real quartet superfield $q^{\,i\,a}$ subjected to the constraints [17]

$$D^{(k\,\underline{k}}q^{\,i)\,a} = D^{(b\,\underline{b}}q^{\,k\,a)} = 0\,, \tag{2.13}$$

where symmetrization is understood for the doublet indices of the same $SU(2)$ group.

In the superspace $\mathbf{HR}^{(1+2+2|8)}$ the same multiplet $(\mathbf{4}, \mathbf{8}, \mathbf{4})$ can be described by a real analytic $\mathcal{N}$=8 superfield $q^{1,1}(\zeta, u, v)$ subjected to the harmonic constraints

$$D^{2,0}q^{1,1} = 0\,, \quad D^{0,2}q^{1,1} = 0\,, \tag{2.14}$$

which in the central basis imply

$$q^{1,1} = q^{i\,a} u^1_i v^1_a\,. \tag{2.15}$$

Then q^{ia} satisfies the constraints (2.13) as a consequence of the constraints (2.9)

$$D^{1,0\,\underline{i}}\, q^{1,1} = D^{0,1\,\underline{a}}\, q^{1,1} = 0\,. \tag{2.16}$$

The analytic basis solution of the harmonic constraints (2.14) is given by

$$\begin{aligned} q^{1,1} \;&=\; f^{i\,a} u^1_i v^1_a + \theta^{1,0\,\underline{i}} \psi^a_{\underline{i}} v^1_a + \theta^{0,1\,\underline{a}} \psi^i_{\underline{a}} u^1_i - i\,(\theta^{1,0})^2 \partial_t f^{i\,a} u^{-1}_i v^1_a - i\,(\theta^{0,1})^2 \partial_t f^{i\,a} u^1_i v^{-1}_a \\ &+\; \theta^{1,0\,\underline{i}} \theta^{0,1\,\underline{a}} F_{\underline{i}\,\underline{a}} - i\,\theta^{1,0\,\underline{i}} (\theta^{0,1})^2 \partial_t \psi^a_{\underline{i}} v^{-1}_a - i\,\theta^{0,1\,\underline{a}} (\theta^{1,0})^2 \partial_t \psi^i_{\underline{a}} u^{-1}_i \\ &-\; (\theta^{1,0})^2 (\theta^{0,1})^2\, \partial^2_t f^{i\,a} u^{-1}_i v^{-1}_a \end{aligned} \tag{2.17}$$

where $(\theta^{1,0})^2 = \theta^{1,0\,\underline{k}}\, \theta^{1,0}_{\underline{k}}$, $(\theta^{0,1})^2 = \theta^{0,1\,\underline{a}}\, \theta^{0,1}_{\underline{a}}$. The independent component d=1 fields f^{ia}, $\psi^a_{\underline{i}}$, $\psi^i_{\underline{a}}$ and $F_{\underline{i}\,\underline{a}}$ form the $\mathcal{N}$=8 off-shell multiplet $(\mathbf{4}, \mathbf{8}, \mathbf{4})$.

The general *off-shell* action of n such superfields $q^{1,1\,M}$ $(M = 1, 2, ...n)$ can be written as

$$S^{gen} = \int \mu^{(-2,-2)}\, \mathcal{L}^{2,2}(q^{1,1\,M}, u, v) \tag{2.18}$$

where

$$\mu^{(-2,-2)} = dtdu\, dv\, d^2\theta^{1,0}\, d^2\theta^{0,1} \tag{2.19}$$

is the analytic superspace integration measure normalized as

$$\int d^2\theta^{1,0}\, d^2\theta^{0,1} (\theta^{1,0})^2 (\theta^{0,1})^2 = 1\,.$$

The analytic superfield Lagrangian $\mathcal{L}^{2,2}$ can bear an arbitrary dependence on its arguments, the only restriction being a compatibility with its external charges $(2, 2)$. The free action is

$$S^{free} = \int \mu^{(-2,-2)}\, q^{1,1\,M} q^{1,1\,M}\,. \tag{2.20}$$

Using (2.17) , one finds the component form of the action (2.20) (for one $q^{1,1}$)

$$S^{free} = \frac{1}{2} \int dt \Big\{ \partial_t f^{i\,a}\, \partial_t f_{i\,a} + \frac{i}{2} \Big(\psi^{\underline{i}\,a}\, \partial_t \psi_{\underline{i}\,a} + \psi^{i\,\underline{a}}\, \partial_t \psi_{i\,\underline{a}} \Big) + \frac{1}{4}\, F^{\underline{i}}_{\underline{a}} F^{\underline{a}}_{\underline{i}} \Big\}\,. \tag{2.21}$$

The bosonic part of the general action (2.18) for one $q^{1,1}$ reads

$$S^{n=1}_{bos} = \frac{1}{2} \int dt \Big\{ G(f)\, \partial_t f^{i\,a}\, \partial_t f_{i\,a} + \frac{1}{4}\, G(f)\, F^{\underline{i}}_{\underline{a}} F^{\underline{a}}_{\underline{i}} \Big\} \tag{2.22}$$

where

$$\begin{aligned} &G(f) = \int du\, dv\, g(f^{1,1}\,, u\,, v)\,, \quad g(f^{1,1}\,, u\,, v) = \left.\frac{\partial^2 \mathcal{L}^{2,2}}{\partial q^{1,1}\, \partial q^{1,1}}\right|_{\theta=0}\,, \\ &q^{1,1}\Big|_{\theta=0} = f^{1,1} = f^{i\,a}(t) u^1_i v^1_a\,. \end{aligned} \tag{2.23}$$

After eliminating the auxiliary field by its equation of motion

$$F^{\underline{k}}_{\underline{b}} = 0 \tag{2.24}$$

(modulo some fermionic terms) one obtains the on–shell form of the action (2.22)

$$S^{gen} = \frac{1}{2}\int dt\, G(f)\,\partial_t f^{i\,a}\,\partial_t f_{i\,a}\,. \tag{2.25}$$

The function $G(f)$ defined in (2.23) satisfies the four–dimensional Laplace equation

$$\triangle G(f) = 0\,, \quad \triangle = \frac{\partial^2}{\partial f^{i\,a}\,\partial f_{i\,a}} \tag{2.26}$$

which follows from the definition of $G(q)$.

For any number of the $(\mathbf{4},\mathbf{8},\mathbf{4})$ multiplets one deals with the same bosonic target HKT (hyper-Kähler with torsion) geometry as in the case of twisted $\mathcal{N}{=}(4,4), d{=}2$ multiplets [16].

3 Nonlinear (4,8,4) supermultiplet

In $\mathcal{N}{=}(4,4)$, $d{=}2$ bi-harmonic superspace the requirement of preserving the flat form of the harmonic derivatives $D^{2,0}$, $D^{0,2}$ uniquely selects the infinite-dimensional "large" $\mathcal{N}{=}4$ superconformal groups (both in the left and right light cone sectors) as the most general coordinate groups meeting this requirement. The same requirement in the $d{=}1$ version of bi-harmonic superspace, together with the demand of covariance of the defining $q^{1,1}$ constraints (2.14), pick up a supergroup which does not coincide with any known $d{=}1$ superconformal group [14]. It is some $\mathcal{N}{=}8$ superextension of the Heisenberg group $\mathbf{h}(2)$ with an operator central charge [13].

On the other hand, one can expect that the multiplet $(\mathbf{4},\mathbf{8},\mathbf{4})$, along the lines of ref. [3, 17], could be treated as the Goldstone multiplet parametrizing the appropriate coset of the $\mathcal{N}{=}8, d{=}1$ superconformal group $OSp(4^*|4)$ [17]. This supergroup admits a realization on the coordinates Z of the standard $\mathcal{N}{=}8, d{=}1$ superspace $\mathbf{R}^{(1|8)}$ and on the constrained superfield $q^{ia}(Z)$ representing the multiplet $(\mathbf{4},\mathbf{8},\mathbf{4})$ in $\mathbf{R}^{(1|8)}$ (see (2.13)). So one can wonder why this superconformal group does not show up in the analytic superspace description of the multiplet $(\mathbf{4},\mathbf{8},\mathbf{4})$, i.e. why it is absent in the set of coordinate transformations preserving the flat form of $D^{2,0}, D^{0,2}$. As explained in [13], the reason is that passing to the bi-harmonic extension of $\mathbf{R}^{(1|8)}$ reduces the general R-symmetry group $SO(8)$ of $\mathbf{R}^{(1|8)}$ down to its subgroup $SO(4)\times SO(4)$, while no $\mathcal{N}{=}8, d{=}1$ superconformal groups with such R-symmetry exist [14]. In particular, R-symmetry subgroup of $OSp(4^*|4)$ is $USp(4)\times SU(2) \sim SO(5)\times SU(2)$. Hence, in the $d{=}1$ bi-harmonic superspace it is impossible to realize any standard $\mathcal{N}{=}8$ superconformal group, under the assumption that the corresponding R-symmetry group acts *linearly* on the harmonic variables $u^{\pm1}_i, v^{\pm1}_a$. It was also noticed in [13] that, given Goldstone $(\mathbf{4},\mathbf{8},\mathbf{4})$ multiplet, with physical bosons parametrizing the R-symmetry coset $SO(5)/SO(4)$, the R-symmetry $SO(5)$ should act on the harmonic variables by the transformations which are *nonlinear* in these physical bosonic fields. This extends to the whole $OSp(4^*|4)$ group which was conjectured to admit a realization in the analytic bi-harmonic superspace, such that the corresponding coordinate variations involve the superfield $q^{1,1}$ itself. Here we show that this hypothesis is true. We explicitly present the $SO(5)/SO(4)$ transformations and transformations of the $\mathcal{N}{=}8$ Poincaré supersymmetry. Surprisingly, the latter are also nonlinear in $q^{1,1}$.

Taking into account that $OSp(4^\star|4)$ involves only three commuting $SU(2)$ groups, the harmonic superspace description of the nonlinear $(\mathbf{4},\mathbf{8},\mathbf{4})$ multiplet, as opposed to the description of the linear one, should respect only three out of four $SU(2)$ symmetries realized in the analytic superspace. To have a correspondence with the linear multiplet, it is natural to assume that two $SU(2)$ acting on the harmonic variables still remain unbroken. So we are led to identify two $SU(2)$ groups acting on the external doublet indices $\underline{i}$ and $\underline{a}$ of the analytic Grassmann coordinates. We also should always keep the balance of the harmonic $U(1)$ charges. The non-linear transformations of the $\mathcal{N}$=8 supersymmetry still preserving the Grassmann bi-harmonic analyticity are then as follows:

$$\begin{aligned}
&\delta t = 2i\left(\epsilon^{-1,0\,\underline{i}}\theta^{1,0}_{\underline{i}} + \epsilon^{0,-1\,\underline{i}}\theta^{0,1}_{\underline{i}}\right), \;\; \delta\theta^{1,0\,\underline{i}} = \epsilon^{1,0\,\underline{i}} + q^{1,1}\epsilon^{0,-1\,\underline{i}}, \;\; \delta\theta^{0,1\,\underline{i}} = \epsilon^{0,1\,\underline{i}} + q^{1,1}\epsilon^{-1,0\,\underline{i}}, \\
&\delta q^{1,1} \simeq q^{1,1\prime}(\zeta', u) - q^{1,1}(\zeta, u) = 0\,, \qquad (3.1)
\end{aligned}$$

which should be compared with (2.6). It is easy to see that these modified $\mathcal{N}$=8 supersymmetry transformations close on the t-translations in the same way as the standard linear $\mathcal{N}$=8 ones. They manifestly preserve the analytic superspace. The analytic measure (2.19) transforms as

$$\delta\mu^{(-2,-2)} = -\left(\partial_{1,0\,\underline{i}}q^{1,1}\epsilon^{0,-1\,\underline{i}} + \partial_{0,1\,\underline{i}}q^{1,1}\epsilon^{-1,0\,\underline{i}}\right)\mu^{(-2,-2)}\,. \qquad (3.2)$$

From this transformation law it follows, in particular, that the analytic superspace integral of any power of $q^{1,1}$ is invariant up to a total derivative in the integrand.

The flat harmonic derivatives (2.11) are obviously not covariant with respect to the transformations (3.1). It is rather straightforward to find their fully covariant analogs

$$\hat{D}^{2,0} = D^{2,0} + q^{1,1}\left(\theta^{1,0\,\underline{i}}\partial_{0,1\,\underline{i}} + q^{1,1}\partial^{0,-2}\right), \quad \hat{D}^{0,2} = D^{0,2} + q^{1,1}\left(\theta^{0,1\,\underline{i}}\partial_{1,0\,\underline{i}} + q^{1,1}\partial^{-2,0}\right), \quad (3.3)$$

where $\partial^{-2,0} = u^{-1}_i\frac{\partial}{\partial u^1_i}$ and $\partial^{0,-2} = v^{-1}_a\frac{\partial}{\partial v^1_a}$. One can check that these operators are invariant under (3.1) up to terms containing $\hat{D}^{2,0}q^{1,1}$ and $\hat{D}^{0,2}q^{1,1}$. Then, as the natural generalization of the linear case constraints (2.14), we are led to impose on $q^{1,1}$ the following constraints:

$$\hat{D}^{2,0}q^{1,1} = \hat{D}^{0,2}q^{1,1} = 0\,. \qquad (3.4)$$

With taking them into account, one easily finds that $\delta\hat{D}^{2,0} = \delta\hat{D}^{2,0} = 0$ under (3.1). One should still check the integrability condition

$$[\hat{D}^{2,0}, \hat{D}^{0,2}]q^{1,1} = 0\,. \qquad (3.5)$$

Commuting the derivatives (3.3) with each other and making use of (3.4), we find

$$[\hat{D}^{2,0}, \hat{D}^{0,2}] = (q^{1,1})^2\left(D^0_u - D^0_v\right), \qquad (3.6)$$

whence (3.5) directly stems.

The constraints (3.4) defining the nonlinear version of the $(\mathbf{4}, \mathbf{8}, \mathbf{4})$ multiplet are off-shell like their linear counterparts (2.14) and go over into the latter, if we rescale $q^{1,1}$ by some contraction parameter and then send this parameter to zero. In this limit (3.3) and (3.1) turn into their standard linear counterparts. In what follows we keep this parameter equal to 1, having in mind that it can be re-introduced at any step by the proper rescaling.

Before solving the constraints (3.4) and constructing invariant actions, let us show that (3.4) are covariant under a special realization of $SO(5)$ transformations. We consider only the transformations belonging to the coset $SO(5)/SO(4)$, with the constant parameters b^{ia}. Defining $b^{\pm 1,\pm 1} = b^{ia}u^{\pm}_i v^{\pm}_a$, $b^{\pm 1,\mp 1} = b^{ia}u^{\pm}_i v^{\mp}_a$, one can check that the requirement of preserving both the Grassmann harmonic analyticity and the constraints (3.4) fixes the corresponding trnsformations, up to a rescaling of b^{ia}, as

$$\delta u^1_i = 2(b^{1,-1}q^{1,1})u^{-1}_i\,,\ \delta v^1_a = 2(b^{-1,1}q^{1,1})v^{-1}_a\,,\ \delta u^{-1}_i = \delta v^{-1}_a = 0\,, \tag{3.7}$$

$$\delta\theta^{1,0\,\underline{i}} = (b^{-1,-1}q^{1,1})\theta^{1,0\,\underline{i}} + b^{1,-1}\theta^{0,1\,\underline{i}}\,,\ \ \delta\theta^{0,1\,\underline{i}} = (b^{-1,-1}q^{1,1})\theta^{0,1\,\underline{i}} + b^{-1,1}\theta^{1,0\,\underline{i}}\,, \tag{3.8}$$

$$\delta t = 2ib^{-1,-1}\,\theta^{0,1\,\underline{i}}\theta^{1,0}_{\underline{i}}\,, \tag{3.9}$$

$$\delta q^{1,1} = b^{1,1} + b^{-1,-1}(q^{1,1})^2\,, \tag{3.10}$$

$$\delta\hat{D}^{2,0} = -2(b^{1,-1}q^{1,1})D^0_u\,, \quad \delta\hat{D}^{0,2} = -2(b^{-1,1}q^{1,1})D^0_v\,. \tag{3.11}$$

Note that, while checking (3.11), one should take into account the constraints (3.4), which, on their own, can be easily checked to be covariant under (3.7) - (3.11). The non-symmetric transformations of the harmonic variables as in (3.7) and those of the covariant harmonic derivatives as in (3.11) are typical for the realizations of superconformal group in the analytic harmonic superspaces [18, 15]. Indeed, the $SO(5)/SO(4)$ transformations (3.7) - (3.11) are a part of the d=1 superconformal group $OSp(4^\star|4)$. The transformation (3.10) contains a shift by $b^{1,1} = b^{ia}u^1_i v^1_a$, which means that four bosonic physical fields in $q^{1,1}$ are Goldstone fields belonging to the coset $SO(5)/SO(4)$. By computing the Lie brackets of the transformations (3.7) - (3.11), we can find the explicit realization of the stability subgroup $SO(4)$ on the analytic coordinates and the Goldstone superfield $q^{1,1}$

$$\begin{aligned}
&\delta u^1_j = \left[\lambda^{ik}u^1_i u^1_k + (\lambda^{ab}v^{-1}_a v^{-1}_b)(q^{1,1})^2\right]u^{-1}_j\,,\ \ \delta u^{-1}_j = 0\,,\\
&\delta v^1_c = \left[\lambda^{ab}v^1_a v^1_b + (\lambda^{ik}u^{-1}_i u^{-1}_k)(q^{1,1})^2\right]v^{-1}_c\,,\ \ \delta v^{-1}_c = 0\,,
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&\delta\theta^{1,0\,\underline{i}} = \lambda^{(ik)}u^{-1}_i u^1_k\,\theta^{1,0\,\underline{i}} + \lambda^{(ab)}v^{-1}_a v^{-1}_b q^{1,1}\theta^{0,1\,\underline{i}}\,,\\
&\delta\theta^{0,1\,\underline{i}} = \lambda^{(ab)}v^{-1}_a v^1_b\,\theta^{0,1\,\underline{i}} + \lambda^{(ik)}u^{-1}_i u^{-1}_k q^{1,1}\theta^{1,0\,\underline{i}}\,,\\
&\delta t = i\lambda^{(ik)}u^{-1}_i u^{-1}_k(\theta^{1,0})^2 + i\lambda^{(ab)}v^{-1}_a v^{-1}_b(\theta^{0,1})^2\,,
\end{aligned} \tag{3.13}$$

$$\delta q^{1,1} = \left(\lambda^{ik}u^1_i u^{-1}_k + \lambda^{ab}v^1_a v^{-1}_b\right)q^{1,1}\,. \tag{3.14}$$

Here $\lambda^{(ik)}$ and $\lambda^{(ab)}$ are the parameter of two mutually commuting $SU(2)$ constituents of $SO(4)$. Note that the Lie brackets of the nonlinear $\mathcal{N}$=8 supersymmetry with these $SO(4)$ transformations are again of the form (3.1), with the proper bracket spinor parameters. The $SO(5)/SO(4)$ transformations (3.7) - (3.10) mix the $\epsilon^{i\underline{i}}$ and $\epsilon^{a\underline{i}}$ supersymmetries.

It is easy to show the covariance of (3.4) under the full superconformal group $OSp(4^\star|4)$. We explicitly present only d=1 conformal boosts with the parameter a

$$\begin{aligned}
&\delta t = at^2\,, \quad \delta\theta^{1,0\,\underline{i}} = at\,\theta^{1,0\,\underline{i}}\,,\ \ \delta\theta^{0,1\,\underline{i}} = at\,\theta^{0,1\,\underline{i}}\,,\\
&\delta u^1_i = -2ia(\theta^{1,0})^2u^{-1}_i\,, \quad \delta v^1_c = -2ia(\theta^{0,1})^2v^{-1}_c\,, \quad \delta u^{-1}_i = \delta v^{-1}_c = 0\,,\\
&\delta\hat{D}^{2,0} = 2ia(\theta^{1,0})^2D^0_u\,, \quad \delta\hat{D}^{0,2} = 2ia(\theta^{0,1})^2D^0_v\,,
\end{aligned} \tag{3.15}$$

$$\delta q^{1,1} = -2ia(\theta^{1,0}\theta^{0,1})\,. \tag{3.16}$$

It is sufficient to check covariance only under these transformations, since the conformal supersymmetry appears in the commutator of the conformal boosts with the $\mathcal{N}$=8, d=1 Poincaré supersymmetry (3.1). Then the covariance under conformal supersymmetry and, hence, the full superconformal group $OSp(4^\star|4)$, follows from the covariance under (3.1) and (3.15), (3.16). In particular, the $SO(5)$ transformations (3.7) - (3.10) and (3.12) - (3.14), as well as one more $SU(2)$ symmetry realized on the doublet indices of the analytic Grassmann coordinates $\theta^{1,0\,\underline{i}}, \theta^{0,1\,\underline{i}}$, are contained in the closure of the conformal and ordinary $\mathcal{N}$=8 supersymmetries. It is instructive to present the transformation law of $q^{1,1}$ under conformal $\mathcal{N}$=8 supersymmetry

$$\delta q^{1,1} = 2i\left[(\eta^{1,0}\theta^{0,1}) + (\theta^{1,0}\eta^{0,1}) + q^{1,1}(\eta^{0,-1}\theta^{0,1}) + q^{1,1}(\theta^{1,0}\eta^{-1,0})\right], \tag{3.17}$$

where $\eta^{\pm 1,0\,\underline{i}} = \eta^{i\underline{i}}u_i^{\pm 1}$, $\eta^{0,\pm 1\,\underline{i}} = \eta^{a\underline{i}}v_a^{\pm 1}$ and $\eta^{i\underline{i}}, \eta^{a\underline{i}}$ are the corresponding Grassmann parameters.

The presence of the shift parts in the transformations (3.16) and (3.17) signals that the generators of both the conformal boosts and conformal $\mathcal{N}$=8 supersymmetry belong to the coset part of the nonlinear realization of $OSp(4^\star|4)$ we are dealing with, like the generators of the $SO(5)/SO(4)$ transformations. Correspondingly, the $SU(2)$ singlet part of the auxiliary field in $q^{1,1}$ and all eight physical fermions are Goldstone fields associated with these generators, in a close analogy to four physical bosons which are Goldstone fields associated with the coset $SO(5)/SO(4)$ generators. The set of transformations which are homogeneously realized on fields includes Poincaré $\mathcal{N}$=8 supersymmetry, $SO(4) \subset SO(5)$, $SU(2)$ which acts on the underlined doublet indices and dilatations. The latter also appear in the closure of Poincaré and conformal supersymmetries and act as the proper rescalings of t and analytic Grassmann coordinates. The superfield $q^{1,1}$ (and, respectively, the physical bosons with which $q^{1,1}$ starts) has the zero dilatation weight. The absence of the dilaton (i.e. a field with a non-zero dilatation weight) among the physical bosonic fields is the indication that it is impossible to construct conformally invariant (and $OSp(4^\star|4)$ invariant) actions for the nonlinear $(\mathbf{4},\mathbf{8},\mathbf{4})$ multiplet, in contradistinction to its linear variant for which scale invariant actions exist [13]. This "no-go" theorem also follows from the fact that the analytic superspace integration measure possesses a non-zero dilatation weight (its dimension is 1) and there is no way to compensate its scale transformation since $q^{1,1}$ has the weight 0. So in order to construct superconformally invariant actions of the nonlinear $q^{1,1}$, one needs at least one extra $\mathcal{N}$=8 multiplet with the dilaton among its components. However, some symmetries from $OSp(4^\star|4)$ can be still preserved and, first of all, the $\mathcal{N}$=8 Poincaré supersymmetry (3.1).

In the next Section we will construct the most general $\mathcal{N}$=8 supersymmetric action of the superfield $q^{1,1}$ defined by the nonlinear constraints (3.4) and study the structure of its bosonic sector. Besides the sigma-model type action we will also construct the general potential-type invariant which yields a non-trivial scalar potential upon eliminating the auxiliary fields.

We close this Section with two comments.

- The nonlinear realization of $OSp(4^\star|4)$ constructed above non-trivially mixes the analytic superspace coordinates and the superfield $q^{1,1}$. The latter can be treated as one more analytic coordinate extending the original analytic bi-harmonic superspace to an invariant coset space of the supergroup $OSp(4^\star|4)$. In this respect the considered system supplies a nice example of the "democracy" between coordinates and fields. In the extended analytic superspace $(t, \theta^{1,0\,\underline{i}}, \theta^{0,1\,\underline{i}}, q^{1,1}, u_k^{\pm 1}, v_c^{\pm 1})$ the original (purely coordinate) analytic superspace (ζ, u, v) specifies a $(1+2+2|4)$ dimensional hypersurface. The Goldstone superfield $q^{1,1}(\zeta, u, v)$ subjected to the covariant conditions (3.4) describes the embedding of this hypersurface into the extended analytic superspace.

- The covariantized harmonic derivatives (3.3) look as a particular case of the harmonic derivatives which are pertinent to $\mathcal{N}$=8, d=1 supergravity [13]. Their form (3.3) corresponds to the particular choice of analytic vielbeins in the $\mathcal{N}$=8 supergravity harmonic derivatives as functions of the constrained superfield $q^{1,1}$. This is an indication that the considered system could be reproduced as a special limit of the $\mathcal{N}$=8, d=1 supergravity.

4 Invariant actions

As already mentioned, the general analytic superspace action

$$S^{nonl}_{gen} = \int \mu^{(-2,-2)}\, \mathcal{L}^{2,2}(q^{1,1}, u, v) \tag{4.1}$$

is invariant, up to a total derivative, with respect to the $\mathcal{N}$=8 supersymmetry transformations (3.1). This follows from the property that $q^{1,1}(\zeta, u, v)$ transforms as a scalar under (3.1) and the transformation rule of the integration measure (3.2). It turns out that, due to the nonlinearity of the $q^{1,1}$ constraints (3.4), even the "free" action

$$S^{nonl}_{2} = \int \mu^{(-2,-2)}\, q^{1,1}q^{1,1} \tag{4.2}$$

yields a non-trivial sigma-model type action, in contradistinction to the case of the linear $(\mathbf{4}, \mathbf{8}, \mathbf{4})$ multiplet. The superfield action (4.2) can be checked to respect the homogeneously realized $SO(4)$ symmetry (3.12) - (3.14), while in the general action (4.1) this symmetry is broken. The action (4.1) and its particular case (4.2) respect the third $SU(2)$ symmetry acting on the underlined indices of the Grassmann coordinates and component fields.

One can construct, as in the case of the linear $(\mathbf{4}, \mathbf{8}, \mathbf{4})$ multiplet [13], one more invariant

$$S^{nonl}_{pot} = \int \mu^{(-2,-2)}\, \theta^{1,0\,\underline{i}}\theta^{0,1\,\underline{k}} C_{\underline{ik}} q^{1,1}\,. \tag{4.3}$$

Despite the explicit presence of Grassmann coordinates, the action (4.3) can be checked to be invariant under the nonlinear transformations (3.1), provided that

$$C_{\underline{ik}} = 2im\, \varepsilon_{\underline{ik}}\,, \tag{4.4}$$

where m is a constant of dimension 1 (the specific normalization has been chosen for further convenience). The proof of invariance is based on representing $\epsilon^{1,0\,\underline{i}} = D^{2,0}\epsilon^{-1,0\,\underline{i}}$, $\epsilon^{0,1\,\underline{i}} = D^{0,2}\epsilon^{0,-1\,\underline{i}}$, integrating by parts with respect to $D^{2,0}$, $D^{0,2}$ and making use of the constraints (3.4). In a similar way, the action (4.3) with the condition (4.4) can be proved to be invariant also under the $SO(4)$ transformations (3.12) - (3.14). The invariance under the third $SU(2)$ acting on the underlined indices is obvious. In components, the invariant (4.3) yields a term linear in an auxiliary field and, as in the case of the linear $(\mathbf{4}, \mathbf{8}, \mathbf{4})$ multiplet, produces scalar potentials after elimination of this field by its equation of motion.

In order to pass to the components in the actions(4.1), (4.2), (4.3) and to reveal the relevant bosonic target metric, we firstly need to solve the constraints (3.4).

4.1 Solving the constraints

In what follows we will limit our consideration to the bosonic sector, with all fermionic fields omitted. So we start with the following component expansion of $q^{1,1}$

$$q^{1,1}(\zeta,u,v) = f^{1,1} + (\theta^{1,0})^2 g^{-1,1} + (\theta^{0,1})^2 g^{1,-1} + \theta^{1,0\,\underline{i}}\theta^{0,1\,\underline{k}} F_{\underline{ik}} + (\theta^{1,0})^2(\theta^{0,1})^2 d^{-1,-1}\,, \tag{4.5}$$

where all component fields are functions of t and harmonics $u_i^{\pm 1}, v_c^{\pm 1}$. The superfield constraints (3.4) amount to a set of harmonic differential equations for these component fields. These equations fix the harmonic dependence of the fields and express them in terms of the $(\mathbf{4},\mathbf{4})$ bosonic fields $f^{ia}(t)$ and $F_{\underline{ik}}(t)$ which form the bosonic subset of the off-shell nonlinear $(\mathbf{4},\mathbf{8},\mathbf{4})$ multiplet. These basic 8 bosonic fields (and 8 fermionic fields in the general case) appear as integration constants in the solutions of the harmonic equations.

The component harmonic equations implied by the constraints (3.4) are as follows:

$$\text{(a)}\ \nabla^{2,0} f^{1,1} = 0\,, \quad \text{(b)}\ \nabla^{0,2} f^{1,1} = 0\,, \tag{4.6}$$

$$\text{(a)}\ \mathcal{D}^{2,0} g^{-1,1} + i\partial_t f^{1,1} - \frac{1}{2} f^{1,1} F = 0\,, \quad \text{(b)}\ \mathcal{D}^{0,2} g^{-1,1} = 0\,, \tag{4.7}$$

$$\text{(a)}\ \mathcal{D}^{2,0} g^{1,-1} = 0\,, \quad \text{(b)}\ \mathcal{D}^{0,2} g^{1,-1} + i\partial_t f^{1,1} - \frac{1}{2} f^{1,1} F = 0\,, \tag{4.8}$$

$$\text{(a)}\ \mathcal{D}^{2,0} F_{\underline{ik}} + \varepsilon_{\underline{ik}} f^{1,1} g^{1,-1} = 0\,, \quad \text{(b)}\ \mathcal{D}^{0,2} F_{\underline{ik}} + \varepsilon_{\underline{ik}} f^{1,1} g^{-1,1} = 0\,, \tag{4.9}$$

$$\text{(a)}\mathcal{D}^{2,0} d^{-1,-1} + i\partial_t g^{1,-1} + \partial^{0,-2} G^{1,1} = 0, \text{(b)}\mathcal{D}^{0,2} d^{-1,-1} + i\partial_t g^{-1,1} + \partial^{-2,0} G^{1,1} = 0 \tag{4.10}$$

where

$$F \equiv \varepsilon^{\underline{ik}} F_{\underline{ik}}\,, \quad G^{1,1} \equiv f^{(1,1)}\left(2g^{1,-1}g^{-1,1} - \frac{1}{4} F^{\underline{ik}} F_{\underline{ik}}\right) \tag{4.11}$$

and

$$\begin{aligned} \nabla^{2,0} &= \partial^{2,0} + (f^{1,1})^2 \partial^{0,-2}\,, \quad \nabla^{0,2} = \partial^{0,2} + (f^{1,1})^2 \partial^{-2,0}\,, \\ \mathcal{D}^{2,0} &= \nabla^{2,0} + 2(f^{1,1}\partial^{0,-2} f^{1,1})\,, \quad \mathcal{D}^{0,2} = \nabla^{0,2} + 2(f^{1,1}\partial^{-2,0} f^{1,1})\,. \end{aligned} \tag{4.12}$$

First we solve eqs. (4.6). One can check that they imply the integrability condition

$$\partial^{2,0}(f^{1,1}\partial^{-2,0} f^{1,1}) = \partial^{0,2}(f^{1,1}\partial^{0,-2} f^{1,1})\,, \tag{4.13}$$

whence it follows that

$$f^{1,1}\partial^{0,-2} f^{1,1} = \partial^{2,0}\varphi\,, \quad f^{1,1}\partial^{-2,0} f^{1,1} = \partial^{0,2}\varphi\,, \tag{4.14}$$

where φ, for the time being, is an arbitrary function of $t, u_i^{\pm 1}$ and $v_a^{\pm 1}$. Defining

$$f^{1,1} = e^{-\varphi} \hat{f}^{1,1}\,, \tag{4.15}$$

one observes that (4.21) and (4.14) together imply the linear equations for $\hat{f}^{1,1}$

$$\partial^{2,0} \hat{f}^{1,1} = \partial^{0,2} \hat{f}^{1,1} = 0 \quad \Rightarrow \quad \hat{f}^{1,1} = f^{ia}(t) u_i^1 v_a^1\,. \tag{4.16}$$

Then (4.14) become equations for expressing φ in terms of f^{ia}

$$\begin{aligned} \partial^{2,0}\varphi &= e^{-2\varphi}(\hat{f}^{1,1}\hat{f}^{1,-1}) - (\hat{f}^{1,1}\hat{f}^{1,1}) e^{-2\varphi}\partial^{0,-2}\varphi\,, \\ \partial^{0,2}\varphi &= e^{-2\varphi}(\hat{f}^{1,1}\hat{f}^{-1,1}) - (\hat{f}^{1,1}\hat{f}^{1,1}) e^{-2\varphi}\partial^{-2,0}\varphi\,. \end{aligned} \tag{4.17}$$

From these equations and the property that φ is neutral it follows that the harmonic variables can appear in φ only through the products $\hat{f}^{1,1}\hat{f}^{-1,-1}$ or $\hat{f}^{1,-1}\hat{f}^{-1,1}$. Because of the relation

$$\hat{f}^{1,1}\hat{f}^{-1,-1} - \hat{f}^{1,-1}\hat{f}^{-1,1} = \frac{1}{2}f^{ia}f_{ia} \equiv \frac{1}{2}f^2 \tag{4.18}$$

only one of these products is independent. Choosing $X = \hat{f}^{1,1}\hat{f}^{-1,-1}$ as the argument of φ we obtain for φ the ordinary first-order differential equation the general (regular in the limit $f^{ia} = 0$) solution of which is

$$f^{1,1} = \frac{2}{1+\sqrt{1+4X}}\hat{f}^{1,1}\,, \quad X = \hat{f}^{1,1}\hat{f}^{-1,-1}\,, \tag{4.19}$$

where we absorbed a harmonic-independent integration constant into rescaling of f^{ia}. It is easy to directly check that (4.19) solves both eqs. (4.6). From (4.19) useful relations follow

$$\begin{aligned} \partial_t f^{1,1} &= \frac{1}{\sqrt{1+4X}}\partial_t \hat{f}^{1,1} - \frac{4}{(1+\sqrt{1+4X})^2\sqrt{1+4X}}(\hat{f}^{1,1})^2\partial_t\hat{f}^{-1,-1}\,, \\ \partial^{0,-2}f^{1,1} &= \frac{1}{\sqrt{1+4X}}\hat{f}^{1,-1}\,, \quad \partial^{-2,0}f^{1,1} = \frac{1}{\sqrt{1+4X}}\hat{f}^{-1,1}\,. \end{aligned} \tag{4.20}$$

The general scheme of solving the remaining harmonic equations (4.7) - (4.10) is as follows. Taking into account that the fields $g^{1,-1}, g^{-1,1}$ and $F_{\underline{ik}}$ have the dimension 1 and $d^{-1,-1}$ dimension 2, one expands them over all possible structures having these dimensions, with the coefficients being functions of the harmonic argument X. Then eqs. (4.7) - (4.10) amount to some first-order inhomogeneous differential equations for these coefficients. The field $F_{\underline{ik}}$ is neutral, so eqs. (4.9) fix it up to the integration constants $A_{(\underline{ik})}(t), A(t)$ which are just the auxiliary fields of the nonlinear $(\mathbf{4},\mathbf{8},\mathbf{4})$ multiplet under consideration. The remaining fields carry at least one negative harmonic $U(1)$ charge, so the corresponding equations fully define them in terms of the fields $f^{ia}(t)$ and $A_{(\underline{ik})}(t), A(t)$.

In this way we obtain

$$F_{(\underline{ik})} = \frac{1}{\sqrt{1+4X}}A_{(\underline{ik})}\,, \quad F = 2A - 8i\frac{1}{(1+\sqrt{1+4X})\sqrt{1+4X}}\,\partial_t\hat{f}^{-1,-1}\hat{f}^{1,1}\,, \tag{4.21}$$

$$g^{-1,1} = \frac{1}{\sqrt{1+4X}}\left(A\hat{f}^{-1,1} - i\partial_t\hat{f}^{-1,1}\right), \quad g^{1,-1} = \frac{1}{\sqrt{1+4X}}\left(A\hat{f}^{1,-1} - i\partial_t\hat{f}^{1,-1}\right). \tag{4.22}$$

The expression for $d^{-1,-1}$ is rather cumbersome since there are quite a few possible structures of dimension 2. It is convenient to represent it as a sum of six terms

$$d^{-1,-1} = \sum_{\alpha=1}^{6} d_\alpha^{-1,-1}\,, \tag{4.23}$$

and to give the solutions separately for each term

$$d_1^{-1,-1} = \frac{1+2f^2}{2(1+4X)^{3/2}}A^2\hat{f}^{-1,-1}\,, \quad d_2^{-1,-1} = \frac{1}{4(1+4X)^{3/2}}\left(A^{(\underline{ik})}A_{(\underline{ik})}\right)\hat{f}^{-1,-1}\,,$$

$$d_3^{-1,-1} = -iA\left[\frac{1}{\sqrt{1+4X}}\partial_t \hat{f}^{-1,-1} - \frac{2}{(1+4X)^{3/2}}\left(\partial_t \hat{f}^{1,-1}\hat{f}^{-1,1} + \partial_t \hat{f}^{-1,1}\hat{f}^{1,-1}\right)\hat{f}^{-1,-1}\right],$$
$$d_4^{-1,-1} = \frac{1}{(1+4X)^{3/2}}\left[(\partial_t \hat{f}^{-1,-1})^2 \hat{f}^{1,1} + \partial_t \hat{f}^{1,-1}\partial_t \hat{f}^{-1,1}\hat{f}^{-1,-1}\right],$$
$$d_5^{-1,-1} = -\frac{1}{\sqrt{1+4X}}\partial_t^2 \hat{f}^{-1,-1}, \quad d_6^{-1,-1} = -\frac{i}{\sqrt{1+4X}}\partial_t A \hat{f}^{-1,-1}. \tag{4.24}$$

4.2 The bosonic sector of the bilinear action

We will start with a sum of the simplest sigma-model type action (4.2) and the potential term (4.3). After doing the integration over θ s, the bosonic part of this action is reduced to

$$S_2^{bos} = \int dtdudv\left(2f^{1,1}d^{-1,-1} + 2g^{-1,1}g^{1,-1} - \frac{1}{4}F^{(\underline{ik})}F_{(\underline{ik})} + \frac{1}{8}F^2 + \frac{i}{2}F\right) \equiv \int dtdudv\mathcal{L}_2. \tag{4.25}$$

Now one should substitute the expressions (4.19), (4.21), (4.22) and (4.24) for the fields in (4.25). The resulting Lagrangian is rather simple

$$\mathcal{L}_2 = \frac{1}{(1+4X)^{3/2}}\left(\partial_t f^{ia}\partial_t f_{ia} - \frac{1}{4}A^{(\underline{ik})}A_{(\underline{ik})} - \frac{1}{2}(1+2f^2)A^2 + 2iA\, f^{ia}\partial_t f_{ia}\right) + imA. \tag{4.26}$$

In writing the last term, we used the obvious property that the second term in the expression for $F(t,u,v)$ in (4.21) yields a total t-derivative after integrating over harmonics.

It remains to compute the bi-harmonic integral

$$I = \int dudv\frac{1}{(1+4X)^{3/2}}. \tag{4.27}$$

It can be reduced, by rescaling $X \to \alpha X$, and differentiating with respect to the parameter α, to the simpler integral

$$I' = \int dudv\sqrt{1+4X},$$

which can be computed, e.g., by expanding the integrand in powers of $X = f^{ia}f^{kb}u_i^1 u_k^{-1}v_a^1 v_b^{-1}$. The answer is

$$I' = \frac{2}{3}\left(1 + \frac{1+2f^2}{1+\sqrt{1+2f^2}}\right), \quad I = \frac{2}{(1+\sqrt{1+2f^2})\sqrt{1+2f^2}}. \tag{4.28}$$

Using this result and eliminating the auxiliary fields in (4.26), we arrive at the following sigma-model type Lagrangian for the physical bosonic fields $f^{ia}(t)$:

$$\begin{aligned}\mathcal{L}_2 \;=\; & \frac{2}{(1+\sqrt{1+2f^2})\sqrt{1+2f^2}}\left[\partial_t f^{ia}\partial_t f_{ia} - \frac{2}{1+2f^2}\left(f^{ia}\partial_t f_{ia}\right)^2\right] \\ & - \frac{m^2}{4}\,\frac{1+\sqrt{1+2f^2}}{\sqrt{1+2f^2}} - \frac{2m}{1+2f^2}\left(f^{ia}\partial_t f_{ia}\right).\end{aligned} \tag{4.29}$$

The last term is a total t-derivative and can be omitted. Note that the second part of the target space metric within square brackets and non-trivial potential term in (4.29) resulted from the elimination of the auxiliary field $A(t)$. Thus, starting from the bilinear superfield action (4.2) and the linear superfield potential term (4.3) we finally arrived at the non-trivial sigma-model Lagrangian (4.29) with some nonlinear scalar potential $\sim m^2$.

Although the metric in (4.29) looks not conformally flat, it takes the conformally flat form in the properly chosen coordinates (like any four-dimensional $SO(4)$ invariant metric, e.g. that on the 4-sphere $SO(5)/SO(4)$). These coordinates are defined by

$$y^{ia} = \frac{2}{1+\sqrt{1+2f^2}}\, f^{ia}\,. \tag{4.30}$$

In these coordinates, the Lagrangian (4.29) takes the simpler form

$$\mathcal{L}_2 = \frac{2}{2+y^2}\,\partial_t y^{ia}\partial_t y_{ia} - m^2\frac{1}{2+y^2}\,. \tag{4.31}$$

It is instructive to compare the pullback metric in (4.29) and (4.31) with the analogous metric on the 4-sphere $SO(5)/SO(4)$. The latter, in the f and y coordinates, takes the form

$$\frac{1}{1+2f^2}\left[(\partial_t f)^2 - \frac{2}{1+2f^2}\,(f\cdot\partial_t f)^2\right] = \frac{4}{(2+y^2)^2}(\partial_t y)^2\,, \tag{4.32}$$

where we used the obvious condensed notation. The sigma model Lagrangian (4.32) enjoys an additional invariance under the nonlinear $SO(5)/SO(4)$ transformations

$$\delta f^{ia} = b^{ia} + 2(b\cdot f)f^{ia} \quad \text{or} \quad \delta y^{ia} = (1-\frac{1}{2}y^2)b^{ia} + (b\cdot y)y^{ia} \tag{4.33}$$

(they can be readily derived from the superfield $SO(5)/SO(4)$ transformation (3.10), (3.7)). Lacking of the full $SO(5)$ invariance in (4.29), (4.31) is just a manifestation of the fact that the superfield actions (4.2), (4.3) do not respect the full superconformal symmetry $OSp(4^\star|4)$, but possess only $\mathcal{N}$=8, d=1 Poincaré supersymmetry and $SO(4)$ R-symmetry. As we shall prove, the S^4 Lagrangian (4.32) cannot be reproduced using the nonlinear $(\mathbf{4},\mathbf{8},\mathbf{4})$ multiplet.

4.3 The general action

After substituting the θ expansion (4.5) and doing the Berezin integral, the bosonic part of the component Lagrangian in the general sigma-model type superfield action (4.1) takes the form

$$L^{gen} = \frac{\partial\mathcal{L}^{2,2}}{\partial f^{1,1}}\, d^{-1,-1} + \frac{\partial^2\mathcal{L}^{2,2}}{\partial f^{1,1}\partial f^{1,1}}\, g^{-1,1}g^{1,-1} - \frac{1}{8}\frac{\partial^2\mathcal{L}^{2,2}}{\partial f^{1,1}\partial f^{1,1}}\left(F^{(\underline{ik})}F_{(\underline{ik})} - \frac{1}{2}F^2\right), \tag{4.34}$$

where $\mathcal{L}^{2,2} = \mathcal{L}^{2,2}(f^{1,1},u,v)$. The action (4.25) is reproduced under the choice $\mathcal{L}^{2,2} = f^{1,1}f^{1,1}$. The potential invariant (4.3) makes the same contributon $2imF$, and we will add it in the end.

As the next step, we substitute the expressions (4.19), (4.21), (4.22) and (4.24) for the harmonic fields in (4.34) and, after doing some algebra and eliminating the auxiliary fields A and $A_{(\underline{ik})}$ by their equations of motion, find the following generalization of the Lagrangian (4.29) (where we took into account also the contribution of the potential term):

$$L^{gen} = \mathcal{F}\left[(\partial_t f)^2 - \frac{2}{1+2f^2}\,(f\cdot\partial_t f)^2\right] - \frac{m^2}{1+2f^2}\,(\mathcal{F})^{-1}\,. \tag{4.35}$$

Here

$$\mathcal{F}(f^{ia}) = \frac{1}{2}\int dudv \frac{1}{1+4X}\left[\frac{\partial^2\mathcal{L}^{2,2}}{\partial f^{1,1}\partial f^{1,1}} - 2\frac{\hat{f}^{-1,-1}}{\sqrt{1+4X}}\frac{\partial\mathcal{L}^{2,2}}{\partial f^{1,1}}\right] = \frac{1}{2}\int dudv\, \frac{\partial^2\mathcal{L}^{2,2}}{\partial\hat{f}^{1,1}\partial\hat{f}^{1,1}}\,, \quad (4.36)$$

and the last equality follows from the relation (4.19). Thus we observe the remarkable fact that both the target space metric and the scalar potential are defined by the same function $\mathcal{F}(f^{ia})$ (4.36) which is a nonlinear generalization of the metric function (2.22) of the linear $(\mathbf{4},\mathbf{8},\mathbf{4})$ multiplet sigma-model action. Using the representation (4.36), after some work one can derive the generalization of the Laplace equation (2.26) to the nonlinear case. It is as follows

$$\left[\Delta + 10(f\cdot\partial_f) + 2(f\cdot\partial_f)(f\cdot\partial_f) + 12\right]\mathcal{F} = 0\,, \quad (4.37)$$

where $\Delta = \frac{\partial^2}{\partial f^{ia}\partial f_{ia}}$ and $(f\cdot\partial_f) = f^{ia}\frac{\partial}{\partial f^{ia}}$. The derivation of (4.37) is essentially based on the fact that, as follows from (4.19), $\mathcal{L}^{2,2}$ does not involve $\hat{f}^{-1,1}$ and $\hat{f}^{1,-1}$ and obeys the condition of the "covariant" independence of $\hat{f}^{-1,-1}$:

$$\frac{\partial\mathcal{L}^{2,2}}{\partial\hat{f}^{-1,-1}} + \frac{4}{(1+\sqrt{1+4X})^2}\,(\hat{f}^{1,1})^2\frac{\partial\mathcal{L}^{2,2}}{\partial\hat{f}^{1,1}} = 0\,, \quad (4.38)$$

which is also a corollary of the relation (4.19). Rescaling f^{ia} and choosing the scale as a contraction parameter, it is easy to see that (4.37) goes over into the four-dimensional Laplace equation (2.26) when this parameter goes to zero. It is straightforward to check that the factor (4.28) provides a particular, $SO(4)$ invariant, solution of (4.37). On the other hand, the factor $1/(1+2f^2)$ appearing in the S^4 sigma model Lagrangian (4.32) is not a solution of (4.37). This means that one cannot find a superfield Lagrangian $\mathcal{L}^{2,2}$ which would give rise to the $SO(5)$ invariant action in the bosonic limit. Hence one cannot construct superconformally invariant action on the basis of the nonlinear $(\mathbf{4},\mathbf{8},\mathbf{4})$ multiplet alone. Note that, by going to the coordinates (4.30), the sigma model part of the action (4.35) can be brought into the conformally flat form, despite the fact that in the case of arbitrary function $\mathcal{L}^{2,2}$ the $SO(4)$ symmetry inherent to the particular action (4.29) is definitely broken.

We point out that *any* choice of the analytic Lagrangian $\mathcal{L}^{2,2}(f,u,v)$ gives the metric function $\mathcal{F}(f^{ia})$ obeying eq. (4.37), like any choice of $\mathcal{L}^{2,2}$ (2.18) in the case of linear $(\mathbf{4},\mathbf{8},\mathbf{4})$ multiplet yields the metric function $G(f)$, eq. (2.22), satisfying the 4-dim Laplace equation (2.26). Thus, in both cases the analytic bi-harmonic Lagrangians $\mathcal{L}^{2,2}$ serve as unconstrained potentials of the relevant geometries. While the target geometry associated with the linear $\mathcal{N}$=8 multiplet $(\mathbf{4},\mathbf{8},\mathbf{4})$ is the same as that associated with its $d=2$ cousin, twisted $\mathcal{N}$=(4,4) multiplet, i.e. the HKT geometry, it is not clear which generalization of this geometry we are facing in the considered case of nonlinear $(\mathbf{4},\mathbf{8},\mathbf{4})$ multiplet. The natural conjecture would be that this is a sort of quaternion-Kähler geometry with torsion. The crucial difference of the nonlinear multiplet from the linear one is that the former seems not to be obtainable by dimensional reduction from higher dimensions, at least we do not know how to accomplish this.

5 Concluding remarks

In this paper we constructed the new multiplet of $\mathcal{N}$=8, d=1 supersymmetry, a nonlinear version of the multiplet $(\mathbf{4},\mathbf{8},\mathbf{4})$ [17], using the manifestly $\mathcal{N}$=8 supersymmetric off-shell approach of the bi-harmonic $\mathcal{N}$=8 superspace [13]. This new multiplet is described, like its linear counterpart, by the analytic harmonic superfield $q^{1,1}(\zeta, u, v)$, this time subjected to nonlinear harmonic constraints. We found the realization of the d=1 superconformal group $OSp(4^\star|4)$ in the analytic bi-harmonic superspace and showed that it non-trivially mixes the coordinates with the superfield $q^{1,1}$. The latter is a Goldstone superfield and it can be regarded as an extra analytic coordinate extending the analytic superspace to an analytic coset supermanifold of the full supergroup $OSp(4^\star|4)$. Though it is impossible to construct a superconformally invariant action of $q^{1,1}$, the actions respecting invariance under $\mathcal{N}$=8, d=1 supersymmetry still exist, and we constructed the most general action of this kind. The bosonic target metric in it satisfies a nonlinear generalization (4.37) of the four-dimensional Laplace equation. The bi-harmonic approach automatically provides a general solution of this equation, with the analytic superfield Lagrangian density as a prepotential (it also fully specifies the scalar potential). It still remains to identify the corresponding geometry. The generic metric is conformally flat in proper coordinates. In the simplest $SO(4)$ invariant case it is some deformation of the metric on $S^4 \sim SO(5)/SO(4)$. It would be interesting to seek some other solutions of eq. (4.37).

As one of the problems for the future study, it is interesting to construct combined actions including both linear and nonlinear $q^{1,1}$ multiplets and to study the corresponding invariant actions and their geometric properties. Another problem is to try to add to the nonlinear $q^{1,1}$ multiplet some extra $\mathcal{N}$=8, d=1 multiplet containing a dilaton among its components (e.g. the multiplets $(\mathbf{5}, \mathbf{8}, \mathbf{3})$ or $(\mathbf{3}, \mathbf{8}, \mathbf{5})$ [19, 17]) and to construct a superconformally invariant action of the nonlinear multiplet $(\mathbf{4}, \mathbf{8}, \mathbf{4})$ in such a way. Some other multiplets of $\mathcal{N}$=8, d=1 supersymmetry, e.g. $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ and $(\mathbf{7}, \mathbf{8}, \mathbf{1})$ [17], also admit a description as constrained bi-harmonic analytic superfields [20]. So one can hope to find out nonlinear analogs of these multiplets, following the lines of this paper.

Note added. When this paper was almost ready for the submission to e-archive, there appeared a work of three authors [21] where basically the same results were obtained from a nonlinear realization of $OSp(4^\star|4)$ on the standard constrained $\mathcal{N}$=8 and $\mathcal{N}$=4, d=1 superfields. In particular, eq. (4.37) was derived. It was also shown that in the variables (4.30) it is reduced to the 4-dim Laplace equation for some new scalar metric function.

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